OPTi's Algorithm for Discreteness Determination

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1 Introduction

OPTi ([6]) is an application program for Macintosh, which visualizes how isometric circles, the Ford region, the limit set, etc. change as one deforms the once punctured torus group using the mouse.

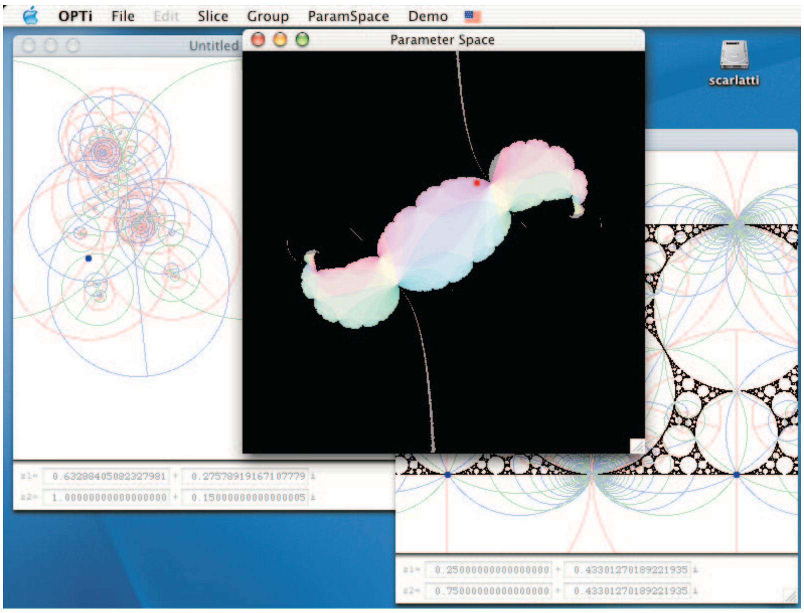

Figure 1: **A screenshot of OPTi**

In this note, we would like to summarize how OPTi draws the parameter space. In the picture of a parameter space, the points corresponding to discrete groups (quasi fuchsian groups) are tiled using various colors, and those corresponding to indiscrete groups are painted out in black. Applying the Jørgensen's inequality to certain sets of generators, the program first tries to decide indiscreteness of the groups, and paints the corresponding points black. If a given group satisfies the Jørgensen's inequality for generators up to a certain depth, the group is considered to have high chance of being discrete. The program then tries to construct the Ford region. When the program succeeds in constructing the Ford region, a fundamental region is obtained, and the Poincaré's Polyhedron Theorem guarantees the discreteness of the group. In that case, the corresponding point is painted with various colors according to the combinatorial pattern of the Ford region.

If the group corresponding to a point in the parameter space is neither decided discrete nor indiscrete, the point remains being gray. There are two kinds of such gray regions. One is a thin gray region sticking out of a cusp. The centerline of such a region contains infinitely many points corresponding to the fundamental groups of cone manifolds. Most of these groups are indiscrete, but the Koebe groups, which are discrete, also reside in this region. The groups in this region are difficult to decide indiscreteness using the Jørgensen's inequality. In the program, you can choose Fine in the ParamSpace menu to apply the Jørgensen's inequality for generators of depth up to 1000, and such gray regions almost disappear.

Another kind of gray region appears in the middle of an area that is considered obviously discrete. The groups in this region require appropriate change of generators before constructing the Ford region. The algorithm in this case is not complete yet.

2 Basics of once punctured torus groups

Here, we collect facts about once punctured torus groups needed later. For the details, the reader is referred to [1, 2].

Let A, B denote the meridian and the longitude of the torus. The fundamental groups of the once punctured torus is then the free group generated by A, B, but we impose the condition that the puncture is a cusp, namely, the commutator $[A, B]$ is parabolic. Let

$$x = \mathrm{tr}\ A, \quad y = \mathrm{tr}\ B, \quad z = \mathrm{tr}\ AB,$$

and the Markoff identity

$$x^2 + y^2 + z^2 = xyz$$

holds. Dividing both sides of the above by xyz, we obtain

$$a_0 + a_1 + a_2 = 1$$

The terms of the left hand side of this equation, namely,

$$a_0 = \frac{x}{yz}, \quad a_1 = \frac{y}{zx}, \quad a_2 = \frac{z}{xy}$$

are called a *complex probability.*

Conversely, given a complex probability (a_0, a_1, a_2), we can restore the original group $\langle A, B \rangle$ as in Fig. 2 (up to conjugacy of groups). Here, P is the Möbius transformation which represents a 180 degree rotation about the geodesic connecting $\pm i/x$ in the upper half "hyperbolic" space. The circle centered at the origin is the

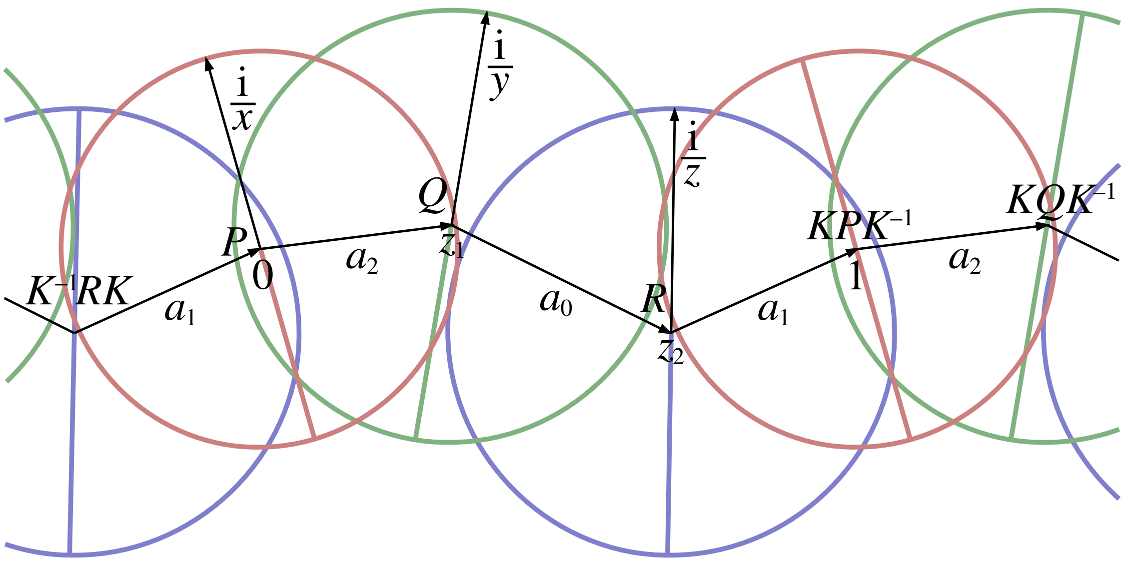

Figure 2: **Restoring the group from a complex probability**

isometric circle of P. We define Q and R similarly. Then $K = RQP$ is a $+1$ translation, and we have $A = RQ = KP$, $B = PQ = K^{-1}R$, $[A, B] = K^2$.

In the above, the triple (P, Q, R) of elements of order 2 is called an elliptic generator triple. It is a set of generators of the fundamental group of the $(2, 2, 2, \infty)$-orbifold, which has the once punctured torus as a branched covering of index 2. Every consecutive triple from the infinite sequence

$$\dots, K^{-1}PK, K^{-1}QK, K^{-1}RK, P, Q, R, KPK^{-1}, KQK^{-1}, KRK^{-1}, \dots$$

is also an elliptic generator triple.

If (P, Q, R) is an elliptic generator triple, (P, R, RQR) is also an elliptic generator triple. This transformation corresponds to a change of marking. We call it simply a change of generators. (Fig. 3) In terms of trace, this operation corresponds to

$$(x, y, z) \longmapsto (x, z, y'), \qquad (y + y' = xz),$$

and in terms of complex probability, to

$$(a_0, a_1, a_2) \longmapsto (a_0 + a_2, \frac{a_0 a_1}{a_0 + a_2}, \frac{a_1 a_2}{a_0 + a_2}).$$

For the infinite sequence obtained from (P, Q, R), there are essentially three different changes of generators, according to which of P, Q, R is omitted. It is known that all the elliptic generators can be obtained by the above operations. Therefore, if we identify elliptic generators belonging to the same infinite sequence and regard them as a point, and regard a change of generators as an edge, we obtain an infinite trivalent tree.

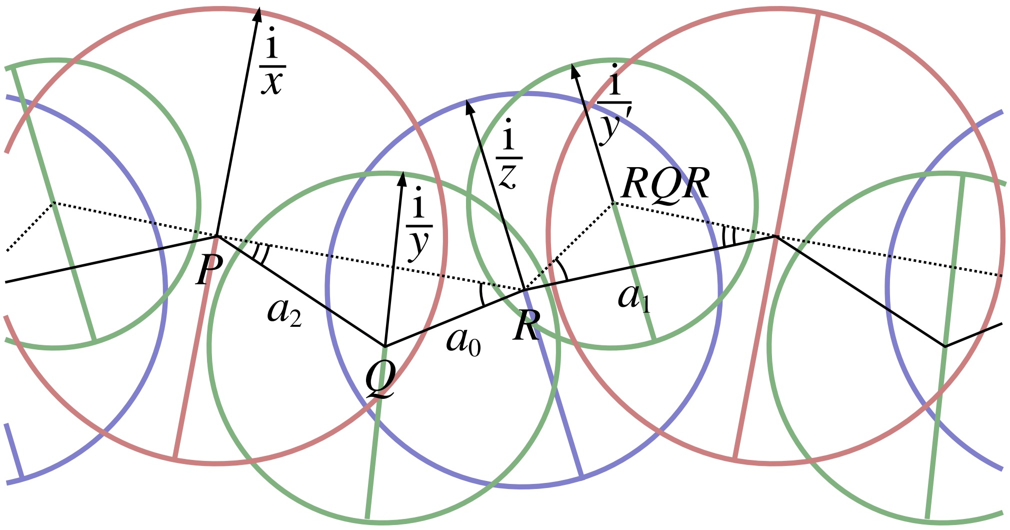

Figure 3: **Change of generators**

3 Indiscreteness determination by the Jørgensen's inequality

The Jørgensen's inequality ([4]) asserts that, if two elements F, G in $SL(2, \mathbf{C})$ generate a non-elementary discrete group, then

$$|\mathrm{tr}^2\ F - 4\,| + |\mathrm{tr}\ [F, G] - 2\,| \geq 1.$$

We apply this for $F = K$, $G = P, Q$, or R. In this case, the condition of being non-elementary always holds, and we obtain a criterion: if the radius of the isometric circle of any one of P, Q, R is greater than 1, then P, Q, R generate an indiscrete group.

The algorithm is: starting from the generators P, Q, R, perform changes of generators successively, and apply this criterion to each newly obtained generator. Checking the criterion for generators of depth up to n may seem to require $3 \cdot 2^n$ checks, but this is not the case.

Computer experiments reveal that, among the three changes of generators applicable to P, Q, R, at least one leads to the situation where the radius of the isometric circle quickly becomes very small as one keeps changing generators, thus there is no need for applying the Jørgensen's inequality in that direction. This fact is supported not only by computer experiments, but also theoretically by Bowditch [5]. Therefore, we can fix a radius, and once the isometric circle of a generator has radius smaller than that, we can stop search in that direction. This allows us to check the Jørgensen's inequality for generators of depth up to n essentially in linear time.

This check is quite effective. The groups not decided indiscrete by this check up to, say, depth 1000 is almost 100% discrete, except for the special case mentioned below: this is the author's feeling acquired through various experiments using OPTi.

In fact, in the ParamSpace menu in OPTi, Coarse, Medium, Fine correspond respectively to check of depth 10, 100, 1000, and while Coarse setting may show some gray region, Medium setting already leaves very little gray region.

Now, let's talk about the special case where the group is indiscrete and yet passes the check by the Jørgensen's inequality. This corresponds to the thin curve-shaped gray region coming out of a cusp in the parameter space shown by OPTi. Choosing points in such a region in OPTi and investigating the groups reveal that, for such a group some successive changes of generators lead to a kind of "rotating pattern", and the radius of the isometric circles becomes neither large nor small. For most of such groups, sufficiently many changes of generators produce an isometric circle of radius greater than 1 after all, and the group is decided to be indiscrete. Namely, the larger we make the depth n of the check by the Jørgensen's inequality, the thinner becomes such a gray region.

However, in the center of such a region, there supposedly lies a 1-dimension worth of groups where some successive changes of generators produce a rotating pattern forever, thus such a region never disappears completely. There are very special cases of these groups where k changes of generators put the generators back onto the same positions. These are the Koebe groups. The Koebe groups are discrete, and satisfy the Jørgensen's inequality. Between the Koebe groups, it is supposed that there lies a 1-dimensional set of groups where changes of generators produce, so to speak, an irrational rotation pattern. But nothing is known about this theoretically.

4 The Ford region for once punctured torus groups

If a group has not been decided to be indiscrete by passing the above check by the Jørgensen's inequality, the program tries construction of the Ford region whose detail will be explained in the next section. Before that, let us give an account of the Jørgensen's theory of the Ford region for once punctured torus groups.

One way of defining the Ford region is, to consider all the isometric hemispheres of the elements of the group not fixing the infinity, and to define it as the boundary pattern (projected down onto the complex plane) of the union of these solid hemispheres. In the case of once punctured torus groups, it is known that we may restrict to the isometric hemispheres of elliptic generators only.

Figure 4 shows an example of a typical once punctured torus group. Solid lines represent the Ford region, and various dotted lines represent complex probabilities. Starting from the lowermost complex probability in the picture, one can see how generators are changed four times to reach the topmost complex probability. Looking at it carefully, one notices that the triangle pattern produced by the complex probabilities is dual to the Ford region in the combinatorial sense.

Jørgensen [3] claims that the Ford region of every geometrically finite once punctured torus group is, as in this example, combinatorially dual to the triangle pattern obtained by applying an appropriate finite changes of generators to some complex probability. We leave the detail of the proof to the forthcoming paper by Akiyoshi,

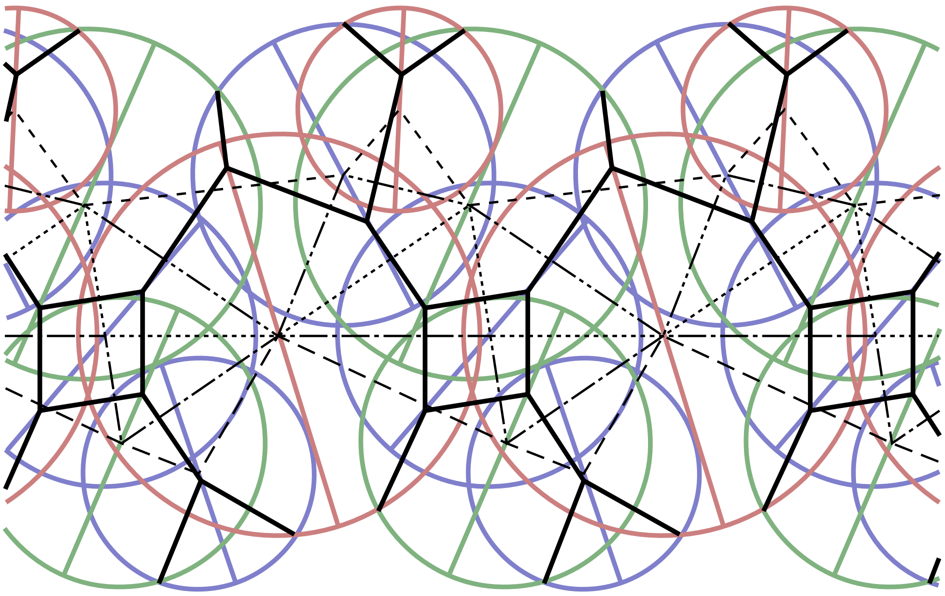

Figure 4: **A typical once punctured torus group**

Sakuma, Wada and Yamashita. Therefore, of the isometric hemispheres of infinitely many sequences of elliptic generators, only the isometric hemispheres of those along a finite path in the infinite tree are “visible from the above”. Let us call these sequences of elliptic generators (and the complex probabilities) corresponding to the Ford region *the Ford sequence*.

Now, according to the Jørgensen’s theory, given a complex probability belonging to the Ford sequence, one can determine which of the three possible changes of generators lead to the upper and lower adjacent complex probabilities as follows.

First, draw all the isometric circles of the elliptic generators corresponding to the given complex probability. If all these circles face the upper region in the complex plane along arcs, the given complex probability corresponds to the upper end of the Ford sequence. (Fig. 5) For each edge of the “line graph” corresponding to the complex probability, consider the triangle defined by the two end points and the upper intersection of the isometric circles of the two generators corresponding to these end points. Then, the above condition is equivalent to the condition that these triangles don’t intersect each other in the interior.

When we draw the triangles along the line graph, if a pair of adjacent triangles intersect each other in the interior, the change of generators which throws away the generator corresponding to the common end points of the adjacent edges leads to the upper adjacent complex probability in the Ford sequence. (Fig. 6)

If two adjacent pairs of triangles both intersect, we need to decide which of the generators to throw away. This can be done as follows. Let a, b, c be consecutive edges of the line graph corresponding to the given complex probability, and assume that both the triangles for a, b and those for b, c intersect each other in the interior.

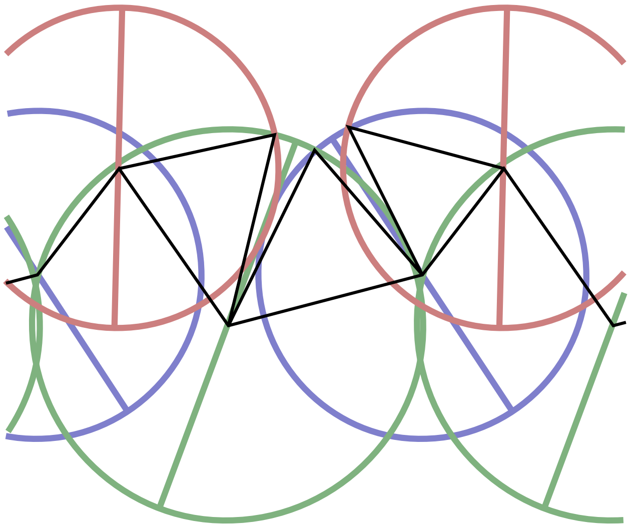

Figure 5: **Upper end of the Ford sequence.** The triangles don't intersect each other.

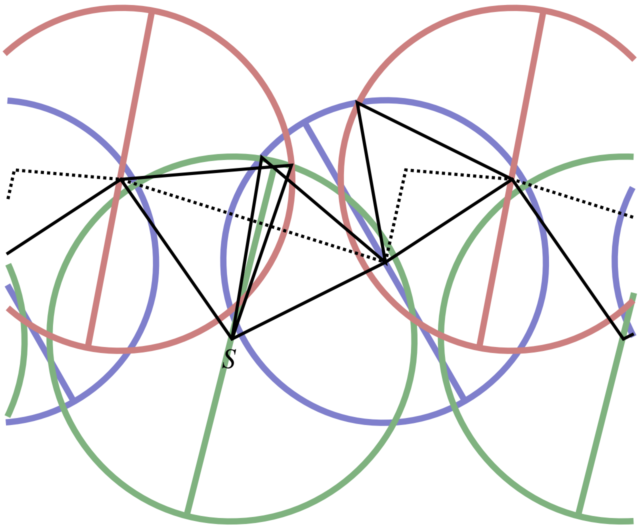

Figure 6: **Case where two triangles intersect.** The change of generators discarding the generator "S" between the two intersecting triangles leads to the upper adjacent complex probability.

(Fig. 7) Draw the line segment α connecting the two intersection points of the isometric circles of the generators corresponding to the two end points of the edge a. Also draw line segments β, γ similarly for the edges b, c. Then, the line segments α and γ both intersect the line segment β in the upper side of the edge b. Now, according to which of α and γ meets β at a closer point to the edge b, the change of generators discarding the generator corresponding to the end point of b on the side of a or c leads to the upper adjacent complex probability in the Ford sequence.

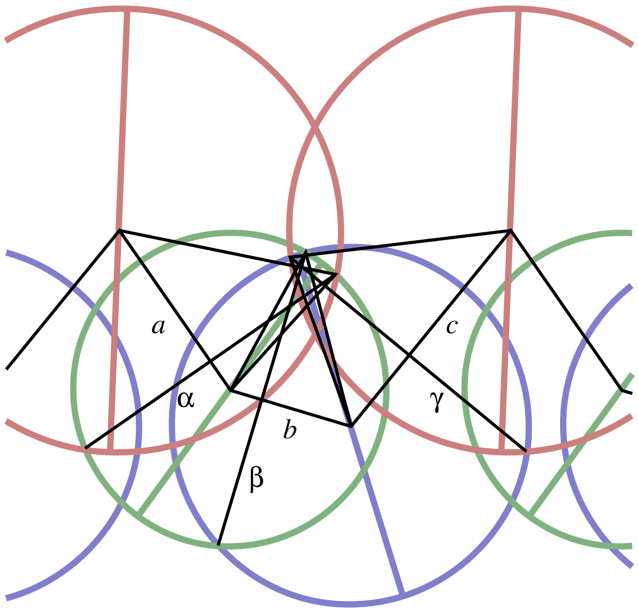

Figure 7: **Case where two pairs of triangles intersect.**

In the above, we have described only about upper adjacent complex probability. The lower adjacent complex probability in the Ford sequence can also be determined by a similar criterion. Therefore, if we have a complex probability in the Ford sequence, we can determine the Ford sequence by computing the upper and lower adjacent complex probabilities one by one, and thus construct the Ford region.

Note that this can only be done under the assumption that a complex probability in the Ford sequence is given. In general, when an arbitrary complex probability is given, no easy way of determining whether it belongs to the Ford sequence is known.

5 Discreteness determination

OPTi tries to determine discreteness as follows.

Given a complex probability, assume as if it belongs to the Ford sequence, and consider computing the upper and lower adjacent complex probabilities as explained in the previous section. We first need to draw the triangles above and below the line graph obtained from the complex probability. There are cases where the triangle inequality fails to hold and we cannot construct the triangles. In such a case, the isometric hemisphere of the generator corresponding to the common end points of two adjacent short edges is completely covered by the isometric hemispheres of the adjacent generators, hence the generator is obviously not a member of the Ford sequence. Therefore we perform the change of generators that discards such a generator.

If the triangle inequality holds everywhere, we construct the triangles above and below and try to compute the upper and lower adjacent complex probabilities. There are cases where first computing the upper adjacent complex probability and then its lower adjacent one does not give the original complex probability. If this happens we know that the original complex probability was not a member of the Ford sequence. In such a case, we throw away all the complex probabilities obtained by then and newly start the algorithm from the point where the inconsistency occurred.

This way we keep constructing a sequence of complex probabilities while paying attention to the consistency mentioned above. If we have obtained a consistent sequence including the upper and lower ends satisfying the criterions for the Ford sequence explained in the previous section, we can show that the complement in the hyperbolic space of the union of the (solid) isometric hemispheres of the generators appearing in the sequence satisfies the conditions of the Poincaré's Polyhedron Theorem, and hence is a fundamental region. Therefore the group is discrete.

Actually, a problematic situation may occur in the above process. Namely, the line graph corresponding to the complex probability appearing in the process may have self-intersections. It is not true that such a complex probability always comes from an indiscrete group. In fact, if we start from a complex probability belonging to the Ford sequence of a discrete group, and perform changes of generators several times in the third direction which does not correspond to the Ford sequence, we often obtain a complex probability of self-intersecting type. Now if we are given such a complex probability of self-intersecting type at the beginning, we know that the group itself is discrete.

It is obvious that a complex probability of self-intersecting type is never a member of the Ford sequence. The question is how we can determine which of the three possible changes of generators leads to the Ford sequence. While Bowditch [5] seems to have a hint for the answer of the question, it is an open question at the moment. To make the point clear, when a complex probability of self-intersecting type appears in the process, OPTi draws the corresponding point gray. The gray regions appearing here and there in the picture of parameter space are such regions.

References

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